Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(01)00069-7
J. Appl. Maths Mechs, Vol. 65, No. 4, pp. 647–664, 2001

0021-8928/01/$—see front matter

PERIODIC MOTIONS OF A SYSTEM CLOSE TO AN AUTONOMOUS REVERSIBLE SYSTEM†

V. N. TKHAI

Moscow

(*Received* 26 *February* 2001)

The problem of the periodic motions of a system containing a small parameter μ and which, in the case of zero value of this parameter, is identical to an autonomous reversible system, is investigated. The periodic motions of an autonomous reversible system form a family. The problem of which of the motions of this family are generating, that is, belong to the family of μ periodic motions and correspond to the value μ = 0, is solved. Both "reversible" perturbations, which preserve the property of reversibility in a perturbed system, as well as perturbations of general form are considered. Both resonance and non-resonance cases and cases where there is an additional "internal" resonance (of the third or fourth order) are studied. The generating solutions, which belong to a Lyapunov family of generating reversible systems as well as a system which is close to a conservative system with one degree of freedom, are investigated in detail. In each of the problems investigated, constructive conditions are obtained for the existence of a periodic motion in the perturbed system. In an application, the dynamics of a Lagrangian gyroscope with a vibrating suspension point is studied. Periodic motions are found in the case of small oscillations of the suspension point and, in particular, the existence of pseudoregular precessions of a gyroscope is established. The cases investigated were omitted from the treatment in earlier papers. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We will consider the problem of periodic motions in the system

$$\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{u}, \mathbf{v}) + \mu\mathbf{U}_1(\mu, \mathbf{u}, \mathbf{v}, t) \tag{1.1}$$

$$\mathbf{v}^{\cdot} = \mathbf{V}(\mathbf{u}, \mathbf{v}) + \mu\mathbf{V}_1(\mu, \mathbf{u}, \mathbf{v}, t); \quad \mathbf{u} \in \mathbb{R}^l, \quad \mathbf{v} \in \mathbb{R}^n \quad (l \geqslant n)$$

$$\mathbf{U}(\mathbf{u}, -\mathbf{v}) = -\mathbf{U}(\mathbf{u}, \mathbf{v}), \quad \mathbf{V}(\mathbf{u}, -\mathbf{v}) = \mathbf{V}(\mathbf{u}, \mathbf{v})$$

with a small parameter μ. When μ = 0, we have a generating or unperturbed system, which is assumed to be reversible and invariant under the replacement of (t, **u**, **v**) by ($-t$, **u**, $-$**v**). For small $|\mu| \neq 0$, we obtain a perturbed system which is close to an autonomous reversible system. The perturbations $\mu\mathbf{U}_1$, $\mu\mathbf{V}_1$ are assumed to be 2π-periodic with respect to the time t.

Periodic motions of an autonomous reversible system always belong to the family in [1]. The question arises as to which of the motions of this family are generating, that is, belong to the family of μ periodic motions and correspond to the value μ = 0. The answer to this question obviously depends on the form of the perturbations which are acting and also on whether there is a resonance or a non-resonance situation. Here, it is natural to assume that the perturbations belong to the class of reversible perturbations which preserve the invariance of the perturbed system under the replacement of (t, **u**, **v**) by ($-t$, **u**, $-$**v**), or to consider perturbations of a more general form when system (1.1) ceases to be reversible when $\mu \neq 0$.

This formulation of the problem is natural and arises in numerous applications, in particular, in mechanics. When the reversible system is a conservative system with one degree of freedom, fairly complete results were obtained in [2, 3].

A problem for a system which is close to a Lyapunov system was previously solved in a similar formulation [4]. The results described below in this paper are close in the scheme of ideas to the results obtained in this problem. Furthermore, an autonomous reversible system, like a Lyapunov system, admits of [5–7] a Lyapunov family of periodic motions. Below, this family is subjected to a careful analysis with the aim of picking out the generating periodic motions.

The applicability of the results obtained below is obviously not restricted by the analysis of a Lyapunov family of reversible systems or conservative systems with one degree of freedom. The basic models which are used in classical and celestial mechanics (the Hill problem, various modifications of the three-body

†*Prikl. Mat. Mekh.* Vol. 65, No. 4, pp. 661–680, 2001.

problem, the N-body problem, a heavy rigid body with a fixed point, a heavy rigid body on an absolutely rough plane, the equations of motion of a mechanical system under the action of positional forces, the equations of motion in quasicoordinates, etc.) are reversible [8, 9] and are described by autonomous equations. Examples, which became canonical examples, are also known of non-local families of periodic motion in these problems, such as elliptic orbits in the two-body problem, a family which contains Grioli precessions, in the problem of the motion of a heavy dynamically asymmetric rigid body around a fixed point and so on.

Finally, we mention that results in the theory of the oscillations of reversible mechanical systems which have been previously obtained [3, 7, 10, 11] are used and developed in this paper. A solution of an alternate problem is given within the framework of the development of a complete theory.

2. ANALYSIS OF A GENERATING SYSTEM

We will assume that, when $\mu = 0$, system (1.1) allows of $T = 2\pi$-periodic motion. The initial value $\mathbf{u}^\circ = \mathbf{u}^*$ and the period $T = 2\pi$ then satisfy the system of n functional equations

$$v_s(\mathbf{u}^\circ, 0, T/2) = 0, \qquad s = 1, \dots, n \tag{2.1}$$

$(\mathbf{u}(\mathbf{u}^\circ, \mathbf{v}^\circ, t), \mathbf{v}(\mathbf{u}^\circ, \mathbf{v}^\circ, t)$ is the solution of the generating system with the initial point $(\mathbf{u}^\circ, \mathbf{v}^\circ)$ when $t = 0)$. This system contains $l + 1$ unknowns $u_1^\circ, \dots, u_1^\circ, T$. Hence, the system of equations (2.1) together with the solution $\mathbf{u}^\circ = \mathbf{u}^*$, $T = 2\pi$ has a k-family of solutions ($k \geqslant l - n$) which leads to the existence, together with the 2π-periodic motion, of a k-family of T-periodic motions containing this motion

$$\mathbf{u} = \boldsymbol{\varphi}(\mathbf{h}, t), \quad \mathbf{v} = \boldsymbol{\psi}(\mathbf{h}, t); \quad \boldsymbol{\varphi}(\mathbf{h}, -t) = \boldsymbol{\varphi}(\mathbf{h}, t), \quad \boldsymbol{\psi}(\mathbf{h}, -t) = -\boldsymbol{\psi}(\mathbf{h}, t)$$

($h_1, \dots, h_k$ are the parameters of the family, that is, the components of the vector $\mathbf{h}$). This family consists of motions which are symmetrical with respect to the fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = 0\}$ of the generating system. In the general situation, the period $T(h_1, \dots, h_k)$ also depends on the parameters of the family and $T(h_1^*, \dots, h_k^*) = 2\pi$.

The variational equations have $k + 1$ solutions of the form

$$\frac{\partial \boldsymbol{\varphi}(h_1, \dots, h_k, t)}{\partial t}, \quad \frac{\partial \boldsymbol{\psi}(h_1, \dots, h_k, t)}{\partial t} \tag{2.2}$$

$$\frac{\partial \boldsymbol{\varphi}(h_1, \dots, h_k, t)}{\partial h_j}, \quad \frac{\partial \boldsymbol{\psi}(h_1, \dots, h_k, t)}{\partial h_j}, \quad j = 1, \dots, k \tag{2.3}$$

and, when $h_j = h_j^*, j = 1, \dots, k$, solution (2.2) is 2π-periodic.

The functions

$$\boldsymbol{\varphi}(h_1, \dots, h_k, T/(2\pi)t), \quad \boldsymbol{\psi}(h_1, \dots, h_k, T/(2\pi)t)$$

have a period equal to 2π which is independent of $h_1, \dots, h_k$. Their derivatives with respect to h_j will therefore also be 2π-periodic functions. We calculate these derivatives, marking the substitution of the values of the parameters $h_j = h_j^*$ $(j = 1, \dots, k)$ with a subscript asterisk.

$$\mathbf{p}_j(t) = \left(\frac{\partial \boldsymbol{\varphi}}{\partial h_j}\right)_* + \frac{t}{2\pi}\left(\frac{\partial T}{\partial h_j}\right)_*\left(\frac{\partial \boldsymbol{\varphi}}{\partial t}\right)_*, \quad \mathbf{q}_j(t) = \left(\frac{\partial \boldsymbol{\psi}}{\partial h_j}\right)_* + \frac{t}{2\pi}\left(\frac{\partial T}{\partial h_j}\right)_*\left(\frac{\partial \boldsymbol{\psi}}{\partial t}\right)_*$$
$$j = 1, \dots, k$$

From this, we find the quantities

$$(\partial \boldsymbol{\varphi} / \partial h_j)_*, \quad (\partial \boldsymbol{\psi} / \partial h_j)_*, \quad j = 1, \dots, k \tag{2.4}$$

which will be odd and even functions of t respectively.

The functions (2.4) are the solutions of the system of variational equations. Subject to the condition that $dT(h_1^*, \dots, h_k^*) \neq 0$, we set up from these functions a system of solutions

$$\mathbf{p}_j(t)\left(\frac{\partial T}{\partial h_1}\right)_* - \mathbf{p}_1(t)\left(\frac{\partial T}{dh_j}\right)_*, \quad \mathbf{q}_j(t)\left(\frac{\partial T}{\partial h_1}\right)_* - \mathbf{q}_1(t)\left(\frac{\partial T}{dh_j}\right)_*, \quad j = 2, \ldots, k \tag{2.5}$$

which are 2π-periodic with respect to t.

The solutions (2.5) are symmetrical with respect to the fixed set $\mathbf{M}_1 = \{\delta\mathbf{u}, \delta\mathbf{v} : \delta\mathbf{v} = \mathbf{0}\}$ of the system of variational equations. Together with solution (2.2), which is symmetrical with respect to the set $\mathbf{M}_2 = \{\delta\mathbf{u}, \delta\mathbf{v} : \delta\mathbf{u} = \mathbf{0}\}$, the functions (2.5) form a system of k 2π-periodic solutions. Furthermore, the variational equations have a single increasing solution of the form (2.4) which is symmetrical with respect to the set $\mathbf{M}_1$. All this means that the variational system has $k + 1$ zero characteristic exponents with k groups of solutions such that, after this system has been reduced to a system with constant coefficients, the equations corresponding to the zero exponents take the form

$$\xi_j^{\cdot} = 0 \ (j = 1, \ldots, k-1), \quad \eta_1^{\cdot} = 0, \quad \zeta_1^{\cdot} = \eta_1 \tag{2.6}$$

In this case, we have $\zeta_1 = 0$ in the set $\mathbf{M}_1$. Furthermore, $k \geqslant l - n + 1$ or the variational equations have at least $l - n$ simple zero characteristic exponents [12].

It follows from the form of the Eqs (2.6) that, according to the zero characteristic exponents, we have a crude case [3, 10, 11] in a problem concerning the continuation of symmetrical periodic motion with respect to a parameter in a class of "reversible" perturbations.

The following theorem therefore holds.

Theorem 1. Suppose a generating, autonomous, reversible system obtained from (1.1), when $\mu = 0$ allows of a 2π-periodic motion. This motion then belongs to the k-family of the parameters $h_1, \ldots, h_k$ of the T-periodic motions. If the period $T(h_1, \ldots, h_k)$ depends on $h_1, \ldots, h_k$, $T(h_1^*, \ldots, h_k^*) = 2\pi$, $dT(h_1^*, \ldots, h_k^*) \neq 0$, then the variational equations have at least $k + 1$ zero characteristic exponents with k groups of solutions ($k \geqslant l - n + 1$) and these exponents do not prevent the extension of the 2π-periodic motion with respect to the parameter μ, if the perturbations belong to the class of "reversible" perturbations.

Corollary 1. If the variational equations have k 2π-periodic solutions, then, for sufficiently small $|\mu| \neq 0$, the reversible system (1.1) has a 2π-periodic motion which changes into the 2π-periodic motion of the generating system when $\pi = 0$.

Proof. When the above-mentioned conditions are satisfied, we necessarily arrive at a subsystem of the form of (2.6). The remaining equations do not contain zero characteristic exponents.

We have the crude case [3, 10, 11] in the sense of the continuation of the periodic motion with respect to a parameter in the class of "reversible" perturbations.

2. If $n = 1$ in system (1.1), the condition $dT(\mathbf{h}^*) \neq 0$ guarantees the continuation of the 2π-periodic motion of the generating system with respect to the parameter μ in the class of "reversible" perturbations.

Proof. In this case, the variational equations only have zero characteristic exponents. They reduce to the form (2.6) and guarantee the continuation of the symmetric periodic motion with respect to the parameter.

Remark 1. The condition $dT \neq 0$ is a natural condition in the case of non-linear vibrations. The vibrations of a linear system do not possess this property.

2. Note that the system of variational equations can also have other zero characteristic exponents in addition to those shown in Theorem 2.

Example 1. All the motions in a two-body problem are plane. The motion in a plane is described by a fourth-order reversible, autonomous system ($l = n = 2$). This system admits of a two-parameter family of symmetric, periodic (elliptic) orbits and $dT \neq 0$. Consequently, $k = 2$, the variational equations are reversible and there are no less than $k + 1 = 3$ zero characteristic exponents, the total number of these exponents being 4.

3. In Example 1, we find one of the interesting problems in which there is a non-local family of periodic motions of a reversible system.

3. A SYSTEM, CLOSE TO A CONSERVATIVE SYSTEM WITH ONE DEGREE OF FREEDOM

Consider the equation

$$z^{\cdot\cdot} + f(z) = \mu F(\mu, z, z^{\cdot}, t) \tag{3.1}$$

where the function $F(\mu, z, z^{\cdot}, t)$ is 2π-periodic in t. The problem of the existence in (3.1) of a vibrational motion which, when $\mu = 0$, converts into one of the vibrations of the generating system has been investigated in [2, 3] as well as the analogous problem for the case of rotational motions [3]. The case of reversible equation (3.1) which satisfies the conditions $f(-z) = -f(z)$, $F(\mu, -z, z^{\cdot}, -t) = -F(\mu, z, z^{\cdot}, t)$ was investigated separately in [3].

Below, we consider reversible equation (3.1) in which

$$F(\mu, z, -z^{\cdot}, -t) = F(\mu, z, z^{\cdot}, t) \tag{3.2}$$

When $\mu = 0$, we have a conservative system with one degree of freedom, an exhaustive analysis of which is carried out using the phase plane method. We assume that this system admits of a family of vibrational motions (Fig. 1), which obviously can be parametrized by the constant x of the energy integral

$$z^{\cdot 2} + V(z) = x(\text{const}), \quad V(z) = 2\int f(z)dz$$

The period of the vibrations is calculated using the formula

$$T(x) = 2\int_{z_{\min}(x)}^{z_{\max}(x)} \frac{dz}{\sqrt{x - V(z)}}$$

where $z_{\min}(x)$, $z_{\max}(x)$ are the smaller and the larger root of the equation $V(z) = x$, respectively.

The vibrations of the generating equation being considered are symmetrical with respect to the abscissa and system (3.1), (3.2) is reversible with a fixed set $\{z, z^{\cdot}: z^{\cdot} = 0\}$. We therefore derive the following theorem from Corollary 2 of Theorem 1.

Theorem 2. The symmetrical $2\pi k$-periodic ($k \in \mathbb{N}$) vibrational motion of a conservative system with one degree of freedom, for which

$$T(x^*) = 2\pi k / m, \quad m \in \mathbb{N}, \quad dT(x^*) \neq 0$$

is continued with respect to the parameter μ in system (3.1), (3.2).

Example 2. The plane vibrations and rotations of a dynamic, symmetrical satellite in an elliptic orbit under the action of gravitational forces and light pressure are described by the reversible equation [13]

$$z'' - \frac{2e\sin\nu}{1 + e\cos\nu} z' = \frac{c(1+e)^2}{(1+e\cos\nu)^4} \sin z \,|\sin z|$$

(z is the angle between the axis of dynamic symmetry and a fixed straight line in the orbital plane, e is

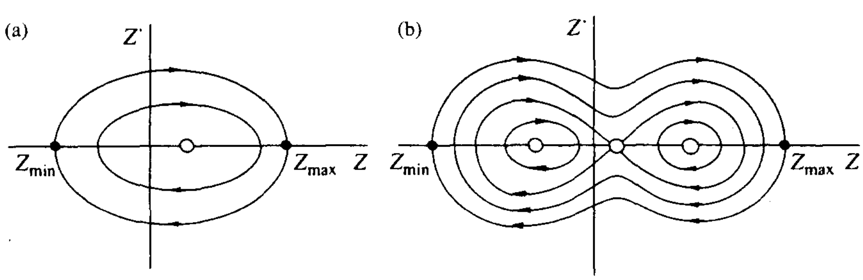

Fig. 1

the eccentricity of the orbit, ν is the true anomaly and c is a parameter which characterizes the light pressure). When $e = 0$ (a circular orbit), we have a smooth conservative system with one degree of freedom

$$z'' = c\sin z\,|\sin z|$$

When $c > 0$, the equilibrium positions $\pm\pi$ are stable and are surrounded by periodic motions, and, when $c < 0$, the origin of the coordinates is such an equilibrium. The period of the above-mentioned vibrations depends on the constant x of the energy integral and, for an equilibrium at $\pm\pi$, we have $dT \neq 0$ [13]. Consequently, (Theorem 2), all the $2\pi k$-periodic vibrations of a satellite are "preserved" in a weakly elliptic ($e \ll 0$) orbit.

Note that this result was formulated earlier in [13]. However, here, a theorem [3] has been used which yields the conditions for the existence of periodic motions which are symmetrical about the z'-axis. The motions about a zero equilibrium position ($c < 0$) are of this type. Periodic motions about the equilibria $\pm\pi$ are symmetrical about the z axis and it is necessary to apply Theorem 2 to them.

4. A LYAPUNOV FAMILY OF PERIODIC MOTIONS

As previously, we consider reversible system (1.1) which is now conveniently written in the form

$$\begin{aligned}\mathbf{u}^{\cdot} &= \mathbf{A}\mathbf{v} + \mathbf{U}_0(\mathbf{u},\mathbf{v}) + \mu\mathbf{U}_1(\mu,\mathbf{u},\mathbf{v},t)\\ \mathbf{v}^{\cdot} &= \mathbf{B}\mathbf{u} + \mathbf{V}_0(\mathbf{u},\mathbf{v}) + \mu\mathbf{V}_1(\mu,\mathbf{u},\mathbf{v},t), \quad \mathbf{u}\in\mathbb{R}^l, \quad \mathbf{v}\in\mathbb{R}^n \quad (l \geqslant n)\end{aligned} \tag{4.1}$$

($\mathbf{A}$ and $\mathbf{B}$ are constant matrices and $\mathbf{U}_0$, $\mathbf{V}_0$ are non-linear terms).

Suppose (a) the characteristic equation of the linear part of system (4.1), when $\mu = 0$, has a pair $\pm i\omega$ of pure imaginary roots, (b) among the other roots of this equation, there are no roots which are equal to $\pm ik\omega(k \in \mathbb{N})$ and (c) rank $B = n$. Then [7], system (4.1), when $\mu = 0$, has an $(l - n)$-parameter set of equilibrium positions which belongs to the fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}$ and contains the zero equilibrium, and a one-parameter family of Lyapunov periodic motions adjoins each point of this set.

We will now investigate the question of the existence, in system (4.1), of a 2π-periodic motion when $\mu \neq 0$. For this purpose, we shall utilize the possibility of reducing system (4.1), when condition c is satisfied, to the form [7]

$$\begin{aligned}\boldsymbol{\xi}^{\cdot} &= \mathbf{P}\mathbf{y} + \boldsymbol{\Xi}_0(\boldsymbol{\xi},\mathbf{x},\mathbf{y}) + \mu\boldsymbol{\Xi}_1(\mu,\boldsymbol{\xi},\mathbf{x},\mathbf{y},t)\\ \mathbf{x}^{\cdot} &= \mathbf{J}\mathbf{y} + \mathbf{X}_0(\boldsymbol{\xi},\mathbf{x},\mathbf{y}) + \mu\mathbf{X}_1(\mu,\boldsymbol{\xi},\mathbf{x},\mathbf{y},t)\\ \mathbf{y}^{\cdot} &= \mathbf{x} + \mathbf{Y}_0(\boldsymbol{\xi},\mathbf{x},\mathbf{y}) + \mu\mathbf{Y}_1(\mu,\boldsymbol{\xi},\mathbf{x},\mathbf{y},t), \quad \boldsymbol{\xi}\in\mathbb{R}^{l-n}, \quad \mathbf{x}\in\mathbb{R}^n, \quad \mathbf{y}\in\mathbb{R}^n\end{aligned} \tag{4.2}$$

($\mathbf{P}$ is a constant matrix and $\mathbf{J}$ is a constant Jordan matrix).

The non-resonance case. We will assume that the Jordan matrix $\mathbf{J}$ does not contain eigenvalues which are close to the number $-p^2$, $p \in \mathbb{N}$. In this case, we make the substitution: $(\xi, \mathbf{x}, \mathbf{y}) \to (\varepsilon^\sigma\xi, \varepsilon\mathbf{x}, \varepsilon\mathbf{y})$, $0 < \sigma < 1$. As a result, we obtain the following system.

$$\begin{aligned}\boldsymbol{\xi}^{\cdot} &= \varepsilon^{1-\sigma}\mathbf{P}\mathbf{y} + \varepsilon^{-1}\boldsymbol{\Xi}_0(\varepsilon^\sigma\boldsymbol{\xi},\varepsilon\mathbf{x},\varepsilon\mathbf{y}) + \varepsilon^{-1}\mu\boldsymbol{\Xi}_1(\mu,\varepsilon^\sigma\boldsymbol{\xi},\varepsilon\mathbf{x},\varepsilon\mathbf{y},t)\\ \mathbf{x}^{\cdot} &= \mathbf{J}\mathbf{y} + \varepsilon^{-1}\mathbf{X}_0(\varepsilon^\sigma\boldsymbol{\xi},\varepsilon\mathbf{x},\varepsilon\mathbf{y}) + \varepsilon^{-1}\mu\mathbf{X}_1(\mu,\varepsilon^\sigma\boldsymbol{\xi},\varepsilon\mathbf{x},\varepsilon\mathbf{y},t)\\ \mathbf{y}^{\cdot} &= \mathbf{x} + \varepsilon^{-1}\mathbf{Y}_0(\varepsilon^\sigma\boldsymbol{\xi},\varepsilon\mathbf{x},\varepsilon\mathbf{y}) + \varepsilon^{-1}\mu\mathbf{Y}_1(\mu,\varepsilon^\sigma\boldsymbol{\xi},\varepsilon\mathbf{x},\varepsilon\mathbf{y},t)\end{aligned} \tag{4.3}$$

We put $\varepsilon = \mu^{1/3}$. Then, when $\mu = 0$, we have a linear generating system which does not have roots of the characteristic equation equal to $\pm ip$. Consequently [10, 11], this case is not critical, and, for sufficiently small $|\mu| \neq 0$, system (4.2) has a unique 2π-periodic motion. In system (4.3), the "amplitude" of these motions is of the order of $\varepsilon^{-1}\mu$. Hence, in system (4.2), the periodic motion has an "amplitude" of the order of μ.

Theorem 3. In the non-resonance case, system (4.1), in the case of a sufficiently small $|\mu| \neq 0$, admits of a unique 2π-periodic motion and it has an "amplitude" of the order of μ.

The resonance case. We assume that

$$\omega = p + a\mu^{\sigma}, \quad \sigma \geqslant 2/3 \quad (a = \text{const}) \tag{4.4}$$

In this case, we separate the variables x_1, y_1 in system (4.2) which correspond to the pure imaginary roots. Then, all the variables in the Lyapunov family, apart from ξ, x_1, y_1, can be assumed to be equal to zero with an accuracy which is considered below. Next, we change from the variables x_1, y_1 to the complex conjugate variables

$$z = x_1 + iy_1, \quad \bar{z} = x_1 - iy_1$$

and we reduce the system in the variables $\xi, z, \bar{z}$ to normal form up to terms of the third order inclusive. Then

$$z^{\cdot} = iz[\omega + \omega_1(\boldsymbol{\xi}) + Az\bar{z}] + \ldots$$

(A is a real constant) and the frequency Ω of the periodic motion is calculated using the formula

$$\Omega = \omega + \omega_1(\boldsymbol{\xi}) + Az^{\circ}\bar{z}^{\circ} + \ldots, \quad \omega_1(\boldsymbol{\xi}) = \sum_{j=1}^{l-n} C_j\xi_j + \sum_{j,k=1}^{l-n} C_{jk}\xi_j\xi_k$$

(C_j, C_{jk} are real constants, and z° and $\bar{z}^{\circ}$ are the initial values of z and $\bar{z}$ respectively). The Lyapunov family adjoins the equilibrium $\xi = \xi^*, z = \bar{z} = 0$. We assume that $\Omega(\xi^*, |z^{\circ}|) = p \in \mathbb{N}$. Then, in the case of fixed ξ^*, we have $d\Omega(\xi^*, |z^{\circ}|) \neq 0$ only if $A \neq 0$. The following theorem therefore follows from Theorem 1.

Theorem 4. If, when $\mu = 0$, system (4.1) satisfies requirements a, b and c above and, moreover, the resonance condition (4.4) is satisfied, then a 2π-periodic motion exists in system (4.1) in the case of sufficiently small $|\mu| \neq 0$ and almost always ($A \neq 0$).

5. A SECOND-ORDER SYSTEM. THE RESONANCE CASE

It follows from Theorem 3 that, in the non-resonance case, the "amplitude" of the periodic motion is of the order of the small parameter μ. We will now investigate what is the "amplitude" in the resonance case. For this purpose, without any loss of generality, we consider the second-order system

$$\begin{aligned} x^{\cdot} &= -\omega y + X(x, y) + \mu X_1(\mu, x, y, t) \\ y^{\cdot} &= \omega x + Y(x, y) + \mu Y_1(\mu, x, y, t) \end{aligned} \tag{5.1}$$

In this case, the zero equilibrium of the generating system will be the centre.

We make the following substitution: $(x, y) \to (\varepsilon x, \varepsilon y)$. We then obtain

$$\begin{aligned} x^{\cdot} &= -\omega y + \varepsilon^{-1}X(\varepsilon x, \varepsilon y) + \mu\varepsilon^{-1}X_1(\mu, \varepsilon x, \varepsilon y, t) \\ y^{\cdot} &= \omega x + \varepsilon^{-1}Y(\varepsilon x, \varepsilon y) + \mu\varepsilon^{-1}X_1(\mu, \varepsilon x, \varepsilon y, t) \end{aligned} \tag{5.2}$$

and, in the general case, we have

$$X(\varepsilon x, \varepsilon y) = \varepsilon^3 X_0(\varepsilon, x, y), \quad Y(\varepsilon x, \varepsilon y) = \varepsilon^3 Y_0(\varepsilon, x, y)$$

We put $\varepsilon = \mu^{1/3}$ and write system (5.2) in polar coordinates r, θ ($x = r\cos\theta, y = r\sin\theta$). We obtain the system

$$\begin{aligned} r^{\cdot} &= \varepsilon^2[(X_0 + X_1)\cos\theta + (Y_0 + Y_1)\sin\theta] \\ \theta^{\cdot} &= p + \varepsilon^2 r^{-1}[-(X_0 + X_1)\sin\theta + (Y_0 + Y_1)\cos\theta + ar\varepsilon^{3\sigma-2}] \end{aligned} \tag{5.3}$$

in which, by virtue of the reversibility of the system obtained from (5.1) when $\mu = 0$, the conditions

$$X_0(0, r\cos\theta, r\sin\theta)\cos\theta + Y_0(0, r\cos\theta, r\sin\theta)\sin\theta \equiv 0$$

$$\Theta(r, \theta) \equiv -X_0(0, r\cos\theta, r\sin\theta)\sin\theta + Y_0(0, r\cos\theta, r\sin\theta)\cos\theta = Ar^3 \quad (A = \text{const})$$

are satisfied.

System (5.3) depends on the parameter ε. When $\varepsilon = 0$, (5.3) permits of a symmetrical $2\pi/p$ – periodic motion of the form

$$r = r^* \text{ (const)}, \quad \theta = \theta_*(t); \quad \theta_*(t) = \theta_*^+(t) = pt \quad \text{или} \quad \theta_*(t) = \theta_*^-(t) = pt + \pi$$

Suppose that

$$X_*(t) \equiv X_1(0, 0, 0, t) = a_0/2 + a_1\cos t + b_1\sin t + a_2\cos 2t + b_2\sin 2t + \ldots$$
$$Y_*(t) \equiv Y_1(0, 0, 0, t) = a_0^*/2 + a_1^*\cos t + b_1^*\sin t + a_2^*\cos 2t + b_2^*\sin 2t + \ldots \qquad (5.4)$$

We consider two cases.

1. *"Reversible" perturbations.* In this case in formula (5.4), the expansion of the function $X_*(t)$ only contains odd harmonics and the expansion of the function $Y_*(t)$ only contains even harmonics.

According to the general results obtained earlier in [3], the question of the existence, in reversible system (5.3) when $\varepsilon \neq 0$, of a 2π-periodic motion is solved by the amplitude equation

$$\int_0^\pi [\Theta(r^*, \theta_*(t)) + kar^* - X_*(t)\sin\theta_*(t) + Y_*(t)\cos\theta_*(t)]dt = 0$$

($k = 1$ when $\sigma = \frac{2}{3}$ and $k = 0$ when $\sigma > \frac{2}{3}$). We write this equation in the explicit form

$$f(r^*) \pm (a_p^* - b_p)/2 = 0, \quad f(r^*) \equiv Ar^{*3} + kar^* \qquad (5.5)$$

(the sign in front of the bracket corresponds to the sign in the notation for the function $\theta_*^\pm(t)$).

It can be seen (Fig. 2) that, in the non-degenerate case when $A(a_p^* - b_p) \neq 0$, one of the equations (5.5) always has a single simple positive root. The number of such roots for each of the equations is equal to one or two and the overall number of roots is equal to one or three. A 2π-periodic solution of system (5.3) with an "amplitude" of the order of unity corresponds to each root. On taking account of the scaling which is carried out on passing from system (5.2) to (5.3), we conclude that 2π-periodic motions exist in (5.1) with an "amplitude" of the order of $\mu^{1/3}$.

Theorem 5. In the resonance case (4.4), the non-degenerate reversible system (5.1), for sufficiently small $|\mu| \neq 0$, allows of s symmetrical ($s = 1$ or $s = 3$), 2π-periodic motions of the form

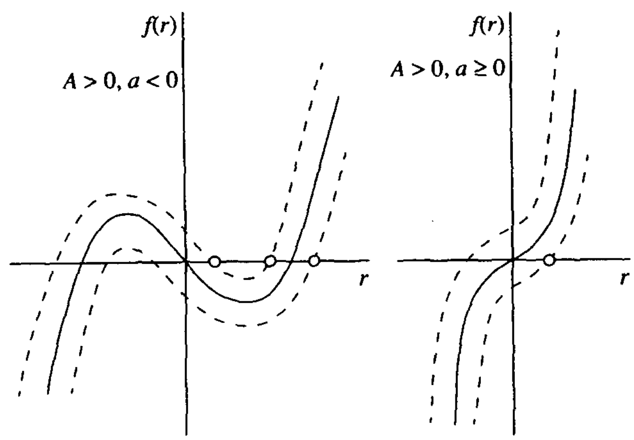

Fig. 2

$$x = \mu^{1/3} r^* \cos pt + o(\mu^{1/3}), \quad y = \mu^{1/3} r^* \sin pt + o(\mu^{1/3})$$

(s is the number of simple positive roots r^* of Eqs (5.5)).

Remark. In the case when $A = 0$, we choose $\varepsilon = \mu^{1/5}$ and take account of the fifth-order terms in the functions X and Y. As a result, we conclude that 2π-periodic motions exist with an "amplitude" of the order of $\mu^{1/5}$ and so on.

2. *Perturbations of general form*. In this case, we have [10] a system of two amplitude equations

$$\int_0^{2\pi} [X_*(t)\cos\theta_*(t) + Y_*(t)\sin\theta_*(t)]dt = 0$$

$$\int_0^{2\pi} [\Theta(r^*, \theta_*(t)) + kar^* - X_*(t)\sin\theta_*(t) + Y_*(t)\cos\theta_*(t)]dt = 0$$

where

$$r = r^*(\text{const}), \quad \theta = \theta_*(t) = pt + \theta^\circ \quad (\theta^\circ = \text{const}) \tag{5.6}$$

is the $2\pi/r$-periodic solution of the generating system obtained from (5.3) when $\varepsilon = 0$.

Substitution of the explicit expressions for X_*, Y_*, Θ leads to the system

$$\begin{aligned} &(a_p + b_p^*)\cos\theta^\circ + (a_p^* - b_p)\sin\theta^\circ = 0 \\ &Ar^{*3} + kar^* + (a_p^* - b_p)\cos\theta^\circ - (a_p + b_p^*)\sin\theta^\circ = 0 \end{aligned} \tag{5.7}$$

It can be seen that, in the non-degenerate case,

$$A[(a_p + b_p^*)^2 + (a_p^* - b_p)^2] \neq 0 \tag{5.8}$$

system (5.7) always has at least one simple solution and the 2π-periodic motion of system (5.1) corresponds to this solution [10]. In particular, when $a_p + b_p^* = 0$, we have $\theta^\circ = 0$ or $\theta^\circ = \pi$, and this solution will be close to a symmetrical solution.

Theorem 6. For a sufficiently small $|\mu| \neq 0$, the non-degenerate resonance system (4.4), (5.1), (5.8) admits of s 2π-periodic solutions of the form

$$x = \mu^{1/3} r^* \cos(pt + \theta^\circ) + o(\mu^{1/3}), \quad y = \mu^{1/3} r^* \sin(pt + \theta^\circ) + o(\mu^{1/3})$$

(s is the number of simple roots (r^*, θ°) of system (5.7) for which $r^* > 0$).

Remark 1. In the analytic case, the generating system in (5.1) is identical to a Lyapunov system and the result of Theorem 6 is known [2].

2. Example 2 is an interesting continuous problem; the theory for a Lyapunov system is inapplicable here.

6. A SYSTEM OF ARBITRARY ORDER. PERTURBATIONS OF GENERAL FORM

For simplicity, we will assume that the condition $\det J \neq 0$ is satisfied in system (4.2). Then, (4.2) reduces to a form in which $\mathbf{P} = 0$. For this purpose, instead of each of the variables ξ_s, it is sufficient to choose the required linear combination of the variables $\xi_1, \ldots \xi_{l-n}, x_1, \ldots, x_n$.

When $\mu = 0$ in (4.2), we have a reversible generating system which allows of a unique family of equilibrium positions

$$\boldsymbol{\xi} = \boldsymbol{\xi}^*(\text{const}), \quad \mathbf{x} = \mathbf{y} = \mathbf{0}$$

We replace ξ by $\xi^* + \xi$ in system (4.2) and, in the resulting system, we change the scale: $(\xi, \mathbf{x}, \mathbf{y}) \to (\varepsilon\xi, \varepsilon\mathbf{x}, \varepsilon\mathbf{y})$. We obtain

$$\begin{aligned} \xi' &= \varepsilon^{-1}\Xi^*(\varepsilon\xi, \varepsilon\mathbf{x}, \varepsilon\mathbf{y}) + \mu\varepsilon^{-1}\Xi_1(\mu, \xi^* + \varepsilon\xi, \varepsilon\mathbf{x}, \varepsilon\mathbf{y}, t) \\ \mathbf{x}' &= [\mathbf{J} + \mathbf{A}(\xi^*)]\mathbf{y} + \varepsilon^{-1}\mathbf{X}^*(\varepsilon\xi, \varepsilon\mathbf{x}, \varepsilon\mathbf{y}) + \mu\varepsilon^{-1}\mathbf{X}_1(\mu, \xi^* + \varepsilon\xi, \varepsilon\mathbf{x}, \varepsilon\mathbf{y}, t) \\ \mathbf{y}' &= [\mathbf{I} + \mathbf{B}(\xi^*)]\mathbf{x} + \varepsilon^{-1}\mathbf{Y}^*(\varepsilon\xi, \varepsilon\mathbf{x}, \varepsilon\mathbf{y}) + \mu\varepsilon^{-1}\mathbf{Y}_1(\mu, \xi^* + \varepsilon\xi, \varepsilon\mathbf{x}, \varepsilon\mathbf{y}, t) \end{aligned} \tag{6.1}$$

(Ξ^*, $\mathbf{X}^*$, $\mathbf{Y}^*$ are non-linear functions, $\Xi^*(\xi, \mathbf{0}, \mathbf{0}) \equiv \mathbf{0}$ and $\mathbf{I}$ is the identity matrix).

We will first consider the non-resonance case when the matrix $\mathbf{C}(\xi^*) = [\mathbf{J} + \mathbf{A}(\xi^*)][\mathbf{I} + \mathbf{B}(\xi^*)]$ does not have eigenvalues close to the number $-p^2 (p \in \mathbb{N})$. Then, on putting $\varepsilon = |\mu|^{1/2}$, we deduce that in (6.1) a 2π-periodic motion $\xi = \text{const}$, $\mathbf{x} = \mathbf{y} = \mathbf{0}$ exists when $\mu \neq 0$. Therefore, when $|\mu| \neq 0$, system (6.1) also has a 2π-periodic motion if the amplitude equation

$$\int_0^{2\pi} \Xi_1(0, \xi^*, 0, 0, t)dt = \mathbf{0} \tag{6.2}$$

has a simple root ξ^*.

Theorem 7. Suppose ξ^* is such a simple root of the amplitude equation (6.2) and the matrix $\mathbf{C}(\xi^*)$ dose not have eigenvalues which are close to the number $-p^2 (p \in \mathbb{N})$. Then, a unique 2π-periodic motion exists in the neighbourhood of the equilibrium position $\xi = \xi^*$, $\mathbf{x} = \mathbf{y} = \mathbf{0}$ of system (4.2) and it has an "amplitude" of the order of μ.

We will now investigate the resonance case (4.4). For this, assuming as before that $\det \mathbf{J} \neq 0$, we put $\mu = 0$ in system (4.2). In the resulting system, we next separate the pair of variables which corresponds to the resonance frequency and write out the quadratic terms of interest in explicit form. As a result, taking into account the property of reversibility of the generating system, we have

$$\begin{aligned} x' &= -\omega y + x\sum_{s=1}^{n-1} a_s^*\zeta_s + y\sum_{s=1}^{n-1} b_s^*\eta_s + \dots \\ y' &= \omega x + x\sum_{s=1}^{n-1} a_s^{**}\zeta_s + y\sum_{s=1}^{n-1} b_s^{**}\eta_s + \dots \\ \xi' &= \mathbf{a}xy + \dots, \quad \boldsymbol{\eta}' = \mathbf{J}_{n-1}\boldsymbol{\xi} + \mathbf{b}xy + \dots, \quad \boldsymbol{\zeta}' = \mathbf{I}_{n-1}\boldsymbol{\eta} + \dots, \quad \boldsymbol{\eta} \in \mathbb{R}^{n-1}, \ \boldsymbol{\zeta} \in \mathbb{R}^{n-1} \end{aligned} \tag{6.3}$$

Here, a_s^*, b_s^*, a_s^{**}, b_s^{**} are real constants, $\mathbf{a}$ and $\mathbf{b}$ are real $(l - n)$- and $(n - 1)$-dimensional vectors respectively, J_{n-1} is an $(n - 1) \times (n - 1)$ Jordan matrix and $\mathbf{I}_{n-1}$ is the $(n - 1) \times (n - 1)$ identity matrix.

Lemma. If the number $-4\omega^2$ is not an eigenvalue of the matrix $\mathbf{J}_{n-1}$, system (6.3) reduces to a form in which all of the coefficients a_s^*, b_s^*, a_s^{**}, b_s^{**} are equal to zero.

The proof is carried out by the direct construction of a polynomial transformation with indefinite constant coefficients. The condition that the coefficients mentioned in the lemma are equal to zero leads to a system of linear equations for determining the coefficients of the transformation. The system is compatible if its determinant $\det(\mathbf{J}_{n-1} + 4\omega^2\mathbf{I}_{n-1}) \neq 0$.

We will now consider the resonance case (4.4) and write system (4.2) in the form

$$\begin{aligned} x' &= -\omega y + X_0(x, y, \xi, \eta, \zeta) + \mu X_1(\mu, x, y, \xi, \eta, \zeta, t) \\ y' &= \omega y + Y_0(x, y, \xi, \eta, \zeta) + \mu Y_1(\mu, x, y, \xi, \eta, \zeta, t) \\ \xi' &= \Xi_0(x, y, \xi, \eta, \zeta) + \mu\Xi_1(\mu, x, y, \xi, \eta, \zeta, t) \\ \eta' &= \mathbf{J}_{n-1}\zeta + \mathbf{H}_0(x, y, \xi, \eta, \zeta) + \mu\mathbf{H}_1(\mu, x, y, \xi, \eta, \zeta, t) \\ \zeta' &= \mathbf{I}_{n-1}\eta + \mathbf{Z}_0(x, y, \xi, \eta, \zeta) + \mu Z_1(\mu, x, y, \xi, \eta, \zeta, t) \end{aligned} \tag{6.4}$$

We shall assume that the transformation which guarantees the lemma has already been carried out in this system. Furthermore, we shall assume that, in system (6.4), the variables x and y are transformed using formulae which ensure normalization of the following second-order system

$$x^{\cdot} = -\omega y + X_0(x,\ y,\ \mathbf{0},\ \mathbf{0},\ \mathbf{0})$$

$$y^{\cdot} = \omega x + Y_0(x,\ y,\ \mathbf{0},\ \mathbf{0},\ \mathbf{0})$$

up to cubic terms inclusive. We now pass into the neighbourhood of the equilibrium

$$x = y = 0,\quad \boldsymbol{\xi} = \boldsymbol{\xi}^*,\quad \boldsymbol{\eta} = \boldsymbol{\zeta} = \mathbf{0}$$

and change the scale by the replacement of (x, y, ξ, η, ζ) by $(\varepsilon x, \varepsilon y, \xi^* + \varepsilon^2\xi, \varepsilon^{5/3}\eta, \varepsilon^{5/3}\zeta)$, $\varepsilon^3 = \mu$. Finally, instead of x and y, we use the polar coordinates r and θ.

As a result of all these transformations, we obtain the following system

$$\begin{aligned}
r^{\cdot} &= \varepsilon^2[(X_0^* + X_1^*)\cos\theta + (Y_0^* + Y_1^*)\sin\theta] \\
\theta^{\cdot} &= p + \varepsilon^2 r^{-1}[-(X_0^* + X_1^*)\sin\theta + (Y_0^* + Y_1^*)\cos\theta + ar\varepsilon^{3\sigma-2}] \\
\boldsymbol{\xi}^{\cdot} &= \varepsilon^{4/3}\boldsymbol{\Xi}_0^* + \varepsilon\boldsymbol{\Xi}_1(\varepsilon^3,\ \varepsilon r\cos\theta,\ \varepsilon r\sin\theta,\ \boldsymbol{\xi}^* + \varepsilon^2\boldsymbol{\xi},\ \varepsilon^{5/3}\boldsymbol{\eta},\ \varepsilon^{5/3}\boldsymbol{\zeta},\ t) \\
\boldsymbol{\eta}^{\cdot} &= \mathbf{J}_{n-1}\boldsymbol{\zeta} + \varepsilon\mathbf{H}_0^* + \varepsilon^{4/3}\mathbf{H}_1(\varepsilon^3,\ \varepsilon r\cos\theta,\ \varepsilon r\sin\theta,\ \boldsymbol{\xi}^* + \varepsilon^2\boldsymbol{\xi},\ \varepsilon^{5/3}\boldsymbol{\eta},\ \varepsilon^{5/3}\boldsymbol{\zeta},\ t) \\
\boldsymbol{\zeta}^{\cdot} &= \mathbf{I}_{n-1}\boldsymbol{\zeta} + \varepsilon\mathbf{Z}_0^* + \varepsilon^{4/3}\mathbf{Z}_1(\varepsilon^3,\ \varepsilon r\cos\theta,\ \varepsilon r\sin\theta,\ \boldsymbol{\xi}^* + \varepsilon^2\boldsymbol{\xi},\ \varepsilon^{5/3}\boldsymbol{\eta},\ \varepsilon^{5/3}\boldsymbol{\zeta},\ t)
\end{aligned} \tag{6.5}$$

Here,

$$\begin{aligned}
&X_0(\varepsilon x,\ \varepsilon y,\ \boldsymbol{\xi}^* + \varepsilon^2\boldsymbol{\xi},\ \varepsilon^{5/3}\boldsymbol{\eta},\ \varepsilon^{5/3}\boldsymbol{\zeta}) = \varepsilon^2 X_0^*(\varepsilon,\ \boldsymbol{\xi}^*,\ x,\ y,\ \boldsymbol{\xi},\ \boldsymbol{\eta},\ \boldsymbol{\zeta}) \\
&Y_0(\varepsilon x,\ \varepsilon y,\ \boldsymbol{\xi}^* + \varepsilon^2\boldsymbol{\xi},\ \varepsilon^{5/3}\boldsymbol{\eta},\ \varepsilon^{5/3}\boldsymbol{\zeta}) = \varepsilon^2 Y_0^*(\varepsilon,\ \boldsymbol{\xi}^*,\ x,\ y,\ \boldsymbol{\xi},\ \boldsymbol{\eta},\ \boldsymbol{\zeta}) \\
&X_1^* = X_1(\varepsilon^3,\ \varepsilon x,\ \varepsilon y,\ \boldsymbol{\xi}^* + \varepsilon^2\boldsymbol{\xi},\ \varepsilon^{5/3}\boldsymbol{\eta},\ \varepsilon^{5/3}\boldsymbol{\zeta},\ t) \\
&Y_1^* = Y_1(\varepsilon^3,\ \varepsilon x,\ \varepsilon y,\ \boldsymbol{\xi}^* + \varepsilon^2\boldsymbol{\xi},\ \varepsilon^{5/3}\boldsymbol{\eta},\ \varepsilon^{5/3}\boldsymbol{\zeta},\ t)
\end{aligned}$$

The functions Ξ_0^*, $\mathbf{H}_0^*$, $\mathbf{Z}_0^*$ vanish when $\varepsilon = 0$ and the expansions of the functions X_1^*, Y_1^* when $\varepsilon = 0$ are given by formulae (5.4) with the sole difference that the coefficients of the expansions now depend on ξ^*. Moreover, when $\varepsilon = 0$, we have

$$X_0^*\cos\theta + Y_0^*\sin\theta \equiv 0,\quad -X_0^*\sin\theta + Y_0^*\cos\theta = Ar^3\ (A = \text{const})$$

When $\varepsilon = 0$, system (6.5) allows of a $2\pi/p$-periodic motion

$$r = r^*(\text{const}),\quad \theta = \theta_*(t) = pt + \theta^0(\theta^0 = \text{const}),\quad \boldsymbol{\xi} = \text{const}$$

In order to solve the problem of the existence of a 2π-periodic motion when $\varepsilon \neq 0$, it is necessary to set up a system of amplitude equations [10]. These equations in the variables r and θ are the same as in system (5.7), and for the variable ξ we obtain

$$\int_0^{2\pi} \boldsymbol{\Xi}_1(0,\ 0,\ 0,\ \boldsymbol{\xi}^*,\ \mathbf{0},\ \mathbf{0},\ t)dt = \mathbf{0} \tag{6.6}$$

Theorem 8. For sufficiently small $|\mu| \neq 0$, the non-degenerate resonance system (4.4), (5.8), (6.4) allows of s 2π-periodic motions of the form

$$x = \mu^{1/3}r^*\cos\theta_*(t) + o(\mu^{1/3}),\quad y = \mu^{1/3}r^*\sin\theta_*(t) + o(\mu^{1/3})$$

$$\boldsymbol{\xi} = \boldsymbol{\xi}^* + \mu\int_0^t \boldsymbol{\Xi}_1^*(\boldsymbol{\xi}^*,\ t)dt + o(\mu),\quad \boldsymbol{\eta} = O(\mu),\quad \boldsymbol{\zeta} = O(\mu)$$

(s is the number of simple roots (r^*, θ^0, ξ^*) of system (5.7), (6.6) for which $r^* > 0$).

7. "INTERNAL" THIRD-ORDER RESONANCE

In the study of the periodic motions of system (4.1) described above, cases of the existence of an "internal" two-frequency resonance, when the characteristic equation, together with a pair of pure imaginary roots $\pm i\omega$, also has the roots $\pm ik\omega (k \in \mathbb{N})$, were excluded from the treatment. We will now return to these cases, taking account of the fact that, in the non-resonance case, the problem is solved by Theorem 3. Here, we will confine ourselves solely to the fourth-order system corresponding to the above-mentioned pure imaginary roots.

In the case of a third-order "internal" resonance, the system in the complex conjugate $z, \bar{z}$ acquires the form

$$\begin{aligned} z_1^{\cdot} &= i\omega_1 z_1 + iB_1\bar{z}_2^2 + Z_{10}(z, \bar{z}) + \mu Z_{11}(\mu, z, \bar{z}, t) \\ z_2^{\cdot} &= -i\omega_2 z_2 + iB_2\bar{z}_1\bar{z}_2 + Z_{20}(z, \bar{z}) + \mu Z_{21}(\mu, z, \bar{z}, t) \end{aligned} \tag{7.1}$$

Here, the generating system has already been reduced to normal form up to the second order inclusive: in system (7.1), B_1 and B_2 are real constant coefficients and the functions Z_{10}, Z_{20} are of no less than the third order with respect to $z, \bar{z}$. Moreover, the complex conjugate group of equations is omitted and the frequency ω_1 is identical to $2\omega_2$, apart from a small parameter.

We change the scale in system (7.1): $(z, \bar{z}) \to (\varepsilon z, \varepsilon\bar{z})$, $\varepsilon^2 = |\mu|$ and henceforth use the polar coordinates r_s, θ_s:

$$z_s = \sqrt{r_s}\exp(i\theta_s), \quad \bar{z}_s = \sqrt{r_s}\exp(-i\theta_s), \quad s = 1, 2$$

As a result, we obtain

$$\begin{aligned} r_\alpha^{\cdot} &= 2\varepsilon B_\alpha r_1^{1/2} r_2 \sin\theta + \varepsilon r_\alpha^{1/2}(Z_\alpha^{*-i\theta_\alpha} + \bar{Z}_\alpha^{*i\theta_\alpha}) \\ \theta_1^{\cdot} &= \omega_1 + \varepsilon B_1 r_1^{-1/2} r_2 \cos\theta + \frac{\varepsilon}{2ir_1^{1/2}}(Z_1^* e^{-i\theta_1} - \bar{Z}_1^* e^{i\theta_1}) \\ \theta_2^{\cdot} &= -\omega_2 + \varepsilon B_2 r_1^{1/2} \cos\theta + \frac{\varepsilon}{2ir_2^{1/2}}(Z_2^* e^{-i\theta_2} - \bar{Z}_2^* e^{i\theta 2}) \end{aligned} \tag{7.2}$$

$$Z_\alpha^* = \varepsilon Z_{\alpha 0}^* + Z_{\alpha 1}^*, \quad Z_{\alpha 0}^* = \varepsilon^{-3} Z_{\alpha 0}(\varepsilon\sqrt{r}\exp(i\theta), \ \varepsilon\sqrt{r}\exp(-i\theta))$$

$$Z_{\alpha 1}^* = Z_{\alpha 1}(\mu, \ \varepsilon\sqrt{r}\exp(i\theta), \ \varepsilon\sqrt{r}\exp(-i\theta), \ t), \quad \alpha = 1, 2; \ \theta = \theta_1 + 2\theta_2$$

Furthermore, if, in system (7.1), we have

$$\begin{aligned} & Z_{s1}(0, 0, 0, t) = X_{s1}^*(t) + iY_{s1}^*(t) \\ & X_{s1}^*(t) \equiv a_{0s}/2 + a_{1s}\cos t + b_{1s}\sin t + a_{2s}\cos 2t + b_{2s}\sin 2t + \dots \\ & Y_{s1}^*(t) \equiv a_{0s}^*/2 + a_{1s}^*\cos t + b_{1s}^*\sin t + a_{2s}^*\cos 2t + b_{2s}^*\sin 2t + \dots \end{aligned} \tag{7.3}$$

then, in system (7.2), we obtain

$$Z_{s1}^{**}(t)e^{-i\theta_s} + \bar{Z}_{s1}^{**}e^{i\theta_s} = 2[X_{s1}^*(t)\cos\theta_s + Y_{s1}^*(t)\sin\theta_s]$$

$$Z_{s1}^{**}(t)e^{-i\theta_s} - \bar{Z}_{s1}^{**}e^{i\theta_s} = 2i[-X_{s1}^*(t)\sin\theta_s + Y_{s1}(t)\cos\theta_s], \quad s = 1, 2$$

$$Z_{s1}^{**}(t) = Z_{s1}^*(0, 0, 0, t) = Z_{s1}(0, 0, 0, t)$$

We consider the resonance case when

$$\omega_1 = p + \alpha_1\mu^\sigma, \quad \omega_2 = p/2 + \alpha_2\mu^\sigma, \quad p \in \mathbb{N}, \quad \alpha_{1,2} = \text{const}, \quad \sigma \geqslant 1 \tag{7.4}$$

Here, when $\varepsilon = 0$, system (7.2) has a $4\pi/p$-periodic solution

$$r_\alpha = r_\alpha^\circ, \quad \alpha = 1, 2; \quad \theta_1 = pt + \theta_1^\circ, \quad \theta_2 = -pt/2 + \theta_2^\circ \tag{7.5}$$

which depends on four arbitrary constants r_α°, θ_α°. When $p = 2q$, $q \in \mathbb{N}$, the solution will be $2\pi/q$-periodic. When $\theta_s^\circ = 0$ or $\theta_s^\circ = \pi$, the above-mentioned solution is symmetrical with respect to the fixed set of the reversible generating system.

According to the previous results [10], simple roots of the system of amplitude equations guarantees the existence of a periodic motion in system (7.1) for sufficiently small $|\mu| \neq 0$. We will now set up these equations in each of the cases being considered.

1. *"Reversible" perturbations*, $p = 2q$, $q \in \mathbb{N}$

$$F \equiv 2B_1 r_2^\circ \cos\theta^\circ + (a_{p1}^* - b_{p1})\cos\theta_1^\circ + 2ka_1\sqrt{r_1^\circ} = 0$$
$$2B_2\sqrt{r_1^\circ r_2^\circ}\cos\theta^\circ + (a_{q2}^* + b_{q2})\cos\theta_2^\circ + 2ka_2\sqrt{r_2^\circ} = 0 \tag{7.6}$$

2. *"Reversible" perturbations*, $p = 2q - 1$, $q \in \mathbb{N}$

$$F = 0, \quad B_2\sqrt{r_1^\circ}\cos\theta^\circ + ka_2 = 0 \tag{7.7}$$

3. *Perturbations of general form*, $p = 2q$, $q \in \mathbb{N}$

$$F_1 \equiv 2B_1 r_2^\circ \sin\theta^\circ + (a_{p1} + b_{p1}^*)\cos\theta_1^\circ + (a_{p1}^* - b_{p1})\sin\theta_1^\circ = 0$$
$$F_2 \equiv 2B_1 r_2^\circ \cos\theta^\circ + (a_{p1}^* - b_{p1})\cos\theta_1^\circ - (a_{p1} + b_{p1}^*)\sin\theta_1^\circ + 2ka_1\sqrt{r_1^\circ} = 0 \tag{7.8}$$
$$2B_2\sqrt{r_1^\circ r_2^\circ}\sin\theta^\circ + (a_{q2} - b_{q2}^*)\cos\theta_2^\circ + (a_{q2}^* + b_{q2})\sin\theta_2^\circ = 0$$
$$2B_2\sqrt{r_1^\circ r_2^\circ}\cos\theta^\circ + (a_{q2}^* + b_{q2})\cos\theta_2^\circ - (a_{q2} - b_{q2}^*)\sin\theta_2^\circ + 2ka_2\sqrt{r_2^\circ} = 0$$

4. *Perturbations of general form*, $p = 2q - 1$, $q \in \mathbb{N}$

$$F_1 = 0, \quad F_2 = 0, \quad B_2\sqrt{r_1^\circ r_2^\circ}\sin\theta^\circ = 0, \quad B_2\sqrt{r_1^\circ}\cos\theta^\circ + ka_2 = 0 \tag{7.9}$$

In formulae (7.6)–(7.9) above, it is necessary to put $k = 1$ when $\sigma = 1/2$ and $k = 0$ when $\sigma > 1/2$. Moreover, $\theta^\circ = \theta_1^\circ + 2\theta_2^\circ$, and, in systems (7.6) and (7.7), we have $\theta_s^\circ = 0$ or $\theta_s^\circ = \pi$.

The following theorem holds.

Theorem 9. A periodic solution

$$z_1 = \mu^{1/2}\sqrt{r_1^\circ}\exp\{i(pt + \theta_1^\circ)\} + o(\mu^{1/2}), \quad z_2 = \mu^{1/2}\sqrt{r_2^\circ}\exp\{i(-pt/2 + \theta_2^\circ)\} + o(\mu^{1/2}) \tag{7.10}$$

of system (7.1) corresponds to each simple root of any of the systems of amplitude equations (7.6)–(7.9). Here, in the cases of (7.6) and (7.7), we have symmetrical periodic motion of the reversible system, in cases (7.6) and (7.8), the solution has a period equal to 2π and, in cases (7.7) and (7.9), the period is equal to 4π.

This assertion follows from the more general results in [10] applied to system (7.1).

Remark. It follows from Theorem 9 that, when $p = 2q - 1$, $q \in \mathbb{N}$ the effect of a doubling of the period in the problem is observed.

We will now analyse systems (7.6)–(7.9) in the case when $k = 0$.

Suppose the notation

$$A_1 = B_1(a_{p1}^* - b_{p1}), \quad A_2 = B_2(a_{q2}^* + b_{q2})$$

is introduced in system (7.6). In the case when $A_1 > 0, A_2 < 0$, we choose $\theta_1^\circ = \pi$, $\theta_2^\circ = 0$ and in the case when $A_1 < 0, A_2 < 0$, we take $\theta_1^\circ = 0$, $\theta_2^\circ = \pi$. Finally, when $A_1 < 0, A_2 < 0$, we put $\theta_1^\circ = \theta_2^\circ = 0$. It is obvious that such a choice of initial angles guarantees the compatibility of system (7.6). It is only in the case when $A_1 > 0, A_2 < 0$ that system (7.6) does not have roots when $k = 0$.

When $k = 0$, system (7.7) always has a simple root (r_1°, r_2°) with $r_1^\circ = 0$, and r_2° are determined from the first equation in (7.7). In this case, the angles θ_1° and θ_2° are chosen such that the quantity $\cos\theta^\circ \cos\theta_1^\circ$ has a sign which is opposite to that of the number A_1.

We now turn to system (7.8). We multiply the first equation by $\sin\theta^\circ$ and add it to the second equation, which has been multiplied by $\cos\theta^\circ$. We obtain the next equation by subtracting the third equation after if has been multiplied by $\sin\theta^\circ$ from the first equation multiplied by $\cos\theta^\circ$. Similar operations are carried out with the second and fourth equations. As result, we obtain the system

$$\begin{aligned}
&2B_1 r_2^\circ + (a_{pl} + b_{pl}^*)\sin 2\theta_2^\circ + (a_{pl}^* - b_{pl})\cos 2\theta_2^\circ = 0\\
&(a_{pl} + b_{pl}^*)\cos 2\theta_2^\circ - (a_{pl}^* - b_{pl})\sin 2\theta_2^\circ = 0\\
&2B_2\sqrt{r_1^\circ r_2^\circ} + (a_{q2} - b_{q2}^*)\sin(\theta_1^\circ + \theta_2^\circ) + (a_{q2}^* + b_{q2})\cos(\theta_1^\circ + \theta_2^\circ) = 0\\
&(a_{q2} - b_{q2}^*)\cos(\theta_1^\circ + \theta_2^\circ) - (a_{q2}^* + b_{q2})\sin(\theta_1^\circ + \theta_2^\circ) = 0
\end{aligned} \tag{7.11}$$

It can be seen that, in the non-degenerate case

$$B_{1,2} \neq 0\ (a_{pl} + b_{pl}^*)(a_{pl}^* - b_{pl}) \neq 0,\ \ (a_{q2} - b_{q2}^*)(a_{q2}^* + b_{q2}) \neq 0$$

system (7.11) always has a simple root $(r_1^\circ, r_2^\circ, \theta_1^\circ, \theta_1^\circ)$.

When $k = 0$, we replace the first and third equations in (7.9) by the first two equations of system (7.11). The existence, when $k = 0$, of a family of θ_1° solutions in system (7.9) then becomes understandable. In this solution $r_1^\circ = 0$.

Note that, when $k \neq 0$, system (7.9) has a simple solution in which $r_1^\circ \neq 0$, $\sin\theta^\circ = 0$.

8. "INTERNAL" FOURTH-ORDER RESONANCE

We will assume that the generating system has been reduced to the normal form up to the third order inclusive. Then, in the complex conjugate variables $z, \bar{z}$, we have the system

$$\begin{aligned}
z_1^\cdot &= i\omega_1 z_1 + iz_1(A_{11}|z_1|^2 + A_{12}|z_2|^2) + iB_1\bar{z}_2^3 + Z_{10}(z, \bar{z}) + \mu Z_{11}(\mu, z, \bar{z}, t)\\
z_2^\cdot &= -i\omega_2 z_2 + iz_2(A_{21}|z_1|^2 + A_{22}|z_2|^2) + iB_2\bar{z}_1\bar{z}_2^2 + Z_{20}(z, \bar{z}) + \mu Z_{21}(\mu, z, \bar{z}, t)
\end{aligned} \tag{8.1}$$

(A_{sj}, B_s are real constants and the functions Z_{s0} are of an order which is no less than the third order in $z, \bar{z}$). As in the treatment of third-order resonance, we change the scale in system (8.1) and now choose $\varepsilon = \mu^{1/3}$. We next write the system in polar coordinates

$$\begin{aligned}
r_\alpha^\cdot &= 2\varepsilon^2 B_\alpha r_1^{1/2} r_2^{3/2}\sin\theta + \varepsilon^2 r_\alpha^{1/2}(Z_\alpha^* e^{-i\theta_\alpha} + \bar{Z}_\alpha^* e^{i\theta_\alpha}),\quad \alpha = 1, 2\\
\theta_1^\cdot &= \omega_1 + \varepsilon^2(A_{11}r_1 + A_{12}r_2 + B_1 r_1^{-1/2} r_2^{3/2}\cos\theta) + \frac{\varepsilon}{2ir_1^{1/2}}(Z_1^* e^{-i\theta_1} - \bar{Z}_1^* e^{i\theta_1})\\
\theta_2^\cdot &= -\omega_2 + \varepsilon^2(A_{21}r_1 + A_{22}r_2 + B_2 r_1^{1/2} r_2^{1/2}\cos\theta) + \frac{\varepsilon}{2ir_2^{1/2}}(Z_1^* e^{-i\theta_2} - \bar{Z}_2^* e^{i\theta_2})\\
\theta &= \theta_1 + 3\theta_2
\end{aligned} \tag{8.2}$$

(the functions Z_α^* have the same meaning as in system (7.2)).

We now consider the resonance case when

$$\omega_1 = p + a_1\mu^\sigma,\ \ \omega_2 = p/3 + a_2\mu^\sigma,\ \ p \in \mathbb{N}\ (a_{1,2} = \text{const},\ \sigma \geqslant 2/3)$$

Here, when $\varepsilon = 0$, system (8.2) has a $6\pi/p$-periodic solution

$$r_\alpha = r_\alpha^\circ,\ \alpha = 1, 2;\ \theta_1 = pt + \theta_1^\circ,\ \theta_2 = -pt/3 + \theta_2^\circ$$

(r_α°, θ_α° are constants) and, when $p = 3q$, $q \in \mathbb{N}$, the solution will be $2\pi/q$-periodic. Moreover, in the symmetric solution, we have $\theta_\alpha^\circ = 0$ or $\theta_\alpha^\circ = \pi$.

We now set up the systems of amplitude equations in each of the possible cases

1. *"Reversible" perturbations*, $p = 3q,\ q \in \mathbb{N}$

$$F \equiv 2(A_{11}r_1^\circ + A_{12}r_2^\circ)r_1^{\circ 1/2} + 2B_1 r_2^{\circ 3/2}\cos\theta^\circ + (a_{p1}^* - b_{p1})\cos\theta_1^\circ + 2ka_1 r_1^{\circ 1/2}$$
$$2(A_{21}r_1^\circ + A_{22}r_2^\circ)r_2^{\circ 1/2} + 2B_2 r_1^{\circ 1/2} r_2^\circ\cos\theta^\circ + (a_{q2}^* + b_{q2})\cos\theta_2^\circ + 2ka_2 r_2^{\circ 1/2} = 0 \quad (8.3)$$

2. *"Reversible" perturbations*, $p \neq 3q,\ q \in \mathbb{N}$

$$F = 0, \quad A_{21}r_1^\circ + A_{22}r_2^\circ + B_2\sqrt{r_1^\circ r_2^\circ}\cos\theta^\circ + ka_2 = 0 \quad (8.4)$$

3. *Perturbations of general form*, $p = 3q,\ q \in \mathbb{N}$

$$F_1 \equiv 2B_1 r_2^{\circ 3/2}\sin\theta^\circ + (a_{p1} + b_{p1}^*)\cos\theta_1^\circ + (a_{p1}^* - b_{p1})\sin\theta_1^\circ = 0$$
$$F_2 \equiv 2(A_{11}r_1^\circ + A_{12}r_2^\circ)r_1^\circ + 2B_1 r_2^{\circ 3/2}\cos\theta^\circ + (a_{p1}^* - b_{p1})\cos\theta_1^\circ - (a_{p1} + b_{p1}^*)\sin\theta_1^\circ + 2ka_1 r_1^{\circ 1/2} = 0$$
$$2B_2 r_1^{\circ 1/2} r_2^\circ\sin\theta^\circ + (a_{q2} - b_{q2}^*)\cos\theta_2^\circ + (a_{q2}^* + b_{q2})\sin\theta_2^\circ = 0 \quad (8.5)$$
$$2(A_{21}r_1^\circ + A_{22}r_2^\circ)r_2^{\circ 1/2} + 2B_2 r_1^{\circ 1/2} r_2^\circ\cos\theta^\circ + (a_{q2}^* + b_{q2})\cos\theta_1^\circ - (a_{q2} - b_{q2}^*)\sin\theta_2^\circ + 2ka_2 r_2^{\circ 1/2} = 0$$

4. *Perturbations of general form*, $p \neq 3q,\ q \in \mathbb{N}$

$$F_1 = 0, \quad B_2 r_1^{\circ 1/2} r_2^\circ \sin\theta^\circ = 0$$
$$F_2 = 0, \quad A_{21}r_1^\circ + A_{22}r_2^\circ + B_2\sqrt{r_1^\circ r_2^\circ}\cos\theta^\circ + ka_2 = 0 \quad (8.6)$$

Theorem 10. A periodic solution

$$z_1 = \mu^{1/3}\sqrt{r_1^\circ}\exp\{i(pt + \theta_1^\circ)\} + o(\mu^{1/3}), \quad z = \mu_2^{1/3}\sqrt{r_2^\circ}\exp\{i(-pt/3 + \theta_2^\circ)\} + o(\mu^{1/3})$$

of system (8.1) corresponds to each simple root of any of the systems of amplitude equations (8.3)–(8.6). Here, in the cases of (8.3) and (8.4), we have a symmetric solution of a reversible system. In cases (8.3) and (8.5), the solution has a period equal to 2π while, in cases (8.4) and (8.6), the period is equal to 6π.

9. THE DYNAMICS OF A LAGRANGIAN GYROSCOPE WITH A VIBRATING SUSPENSION POINT

Consider the motion of a Lagrangian gyroscope (of a dynamically symmetrical rigid body with a centre of mass on the axis of symmetry) about its suspension point O. It is assumed that the point O executes vertical harmonic oscillations $\zeta(t) = a_*$, $\cos\Omega t$ about a certain fixed point.

The equations of motion of the problem are known [14, 15]. If the orientation of the connected system of coordinates is specified by the Euler angles, the coordinates ψ and φ will be cyclic. We put the constant values of the moments p_ψ and p_φ equal to $A\Omega a, A\Omega b$ respectively (A is the equatorial moment of inertia, and a and b are dimensionless constants). We then obtain [14, 15] the following expressions for the angular velocities of the precession and characteristic rotation of the gyroscope.

$$\psi' = \frac{a - b\cos\theta}{\sin^2\theta}, \quad \varphi' = \frac{A}{C}b - \frac{(a - b\cos\theta)\cos\theta}{\sin^2\theta}$$

(C is the axial moment of inertia and a prime denotes differentiation with respect to the dimensionless variable $\tau = \Omega t$).

Hence, the investigation of the motion of the gyroscope reduces to an analysis of a system with one degree of freedom and a generalized coordinate θ

$$\theta'' + \frac{d}{d\theta}\frac{(a-b\cos\theta)^2}{2\sin^2\theta} + (-\alpha + \beta\cos\tau)\sin\theta = 0 \tag{9.1}$$

The dimensionless parameters α and β and defined by the formulae [14, 15]

$$\alpha = \frac{mgz_G}{A\Omega^2}, \quad \beta = \frac{a_*}{l_0}$$

($l_0 = A/(mz_G)$ is the reduced length of the body as a physical pendulum, m is the mass of the gyroscope and Z_G is the distance from the centre of mass **G** to the point **0**). The parameter α ($\alpha > 0$) characterizes the position of the centre of mass on the axis of symmetry and the parameter β ($\beta \geqslant 0$) characterizes the amplitude of the vibration of the suspension point.

When $\beta = 0$, we have a conservative system with one degree of freedom. An exhaustive analysis of this system is carried out using the phase plane method. In particular, all periodic motions can be distinguished using this method.

For small $\beta \neq 0$, we have a system which is close to a conservative system with one degree of freedom. Moreover, this system if reversible. The invariance of system (9.1) with respect to the replacement of (τ, θ) by $(-\tau, \theta)$ can be verified directly.

If follows from what has been stated that the theory developed in the previous sections can be used to investigate the periodic motions of a gyroscope with a small amplitude of vibration of the suspension point. Note that other aspects of this problem have been investigated previously [14]. The case when $|a| \neq |b|$ is considered here and the case when $|\alpha| = |b|$ is analysed below.

The case when $a = b$. Equation (9.1) takes the form

$$\theta'' + \frac{a^2\,\mathrm{tg}(\theta/2)}{2\cos^2(\theta/2)} + (-\alpha + \beta\cos\tau)\sin\theta = 0 \tag{9.2}$$

In the case of a fixed suspension point ($\beta = 0$), we determine all equilibrium positions from the equation

$$\frac{dW(\theta)}{d\theta} \equiv \frac{(a^2 - 4\alpha\cos^4(\theta/2))}{2\cos^3(\theta/2)}\sin(\theta/2) = 0, \quad W(\theta) \equiv \frac{a^2}{2}\mathrm{tg}^2\frac{\theta}{2} + \alpha\cos\theta \tag{9.3}$$

Suppose $a^2 \geqslant 4\alpha$. In this case, we have a unique, zeroth position of equilibrium $\theta_* = 0$ which corresponds to a classical "sleeping" gyroscope. This gyroscope is stable since the condition $a^2 \geqslant 4\alpha$ is identical to the well known Maiyevskii – Chetayev condition $C^2\omega^2 \geqslant 4Amgz_G$ (ω is the angular velocity of the rotation of the gyroscope about the axis of symmetry).

The phase pattern of system (9.2) when $\beta = 0$ is shown in Fig. 3 for the case being considered. We calculate the period of the vibration

$$T_* = 4\int_0^{\theta_0} \frac{d\theta}{\sqrt{2[h - W(\theta)]}}, \quad h = W(\theta_0), \quad 0 < \theta_0 < \pi$$

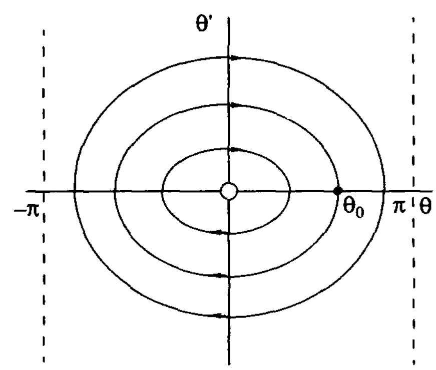

Fig. 3

and use the notation

$$\mathrm{tg}(\theta_0/2)=k,\quad \mathrm{tg}(\theta/2)=ku$$

Then

$$d\theta = 2kdu/(1+k^2u^2)$$

and the expression for the period takes the form

$$T_*(k)=8\int_0^1 \frac{du}{(1+k^2u^2)\sqrt{(1-u^2)[a^2-4\alpha/((1+k^2)(1+k^2u^2))}},\quad T_*(0)=\frac{4\pi}{\sqrt{a^2-4\alpha}} \tag{9.4}$$

It is seen from this that $T_*(k)$ is a strictly decaying function. This can obviously be directly verified by analysing the sign of the derivative.

Graphs of $T(k) = aT_*(k, \gamma)$ are shown in Fig. 4 for different values of $\gamma(\gamma = 4\alpha/a^2)$ with a step size $\Delta\gamma = 0.5$, and in a lower curve $\gamma = -1$.

The condition $dT(k) \neq 0$, together with the property of reversibility of Eq. (9.2), rapidly leads to the conclusion that "conservation" accompanies the small vibrations of the suspension point of the $2\pi s$-periodic motions of a gyroscope for which

$$T_*(k^*)=2\pi s/n,\quad n\in\mathbb{N} \tag{9.5}$$

Among these motions are both 2π-periodic motions ($s = 1$) and motions with a period which is a multiple of 2π. It can be seen from Fig. 4 that, when $a = 1$, 2π-periodic motions occur with an "amplitude" $\theta° = 110° - 130°$ ($1.5 < k < 2.0$) regardless of the value of the parameter γ. The "amplitude" decreases as the parameter a (the angular velocity of the gyroscope) increases.

Local periodic motions, which are close to the rotation of the gyroscope about the vertical, are not observed; only the rotation about the vertical exists.

Suppose $a^2 < 4\alpha$. In this case, we have three equilibrium positions, a zeroth position corresponding to an unstable "sleeping" gyroscope and two symmetrical position of equilibrium with respect to the zero of equilibrium $\pm\theta_*$: $4\alpha\cos^4(\theta_*/2) = a^2$ (Fig. 5).

The period of the vibrations containing all three equilibrium positions is calculated as usual using formula (9.4). This means that all of the $2\pi s$-periodic motions of a gyroscope which satisfy condition (9.5) are "conserved" in the case of small vibrations of the suspension point O.

For the two equilibria $\pm\theta_*$, we calculate

$$\left.\frac{d^2W}{d\theta^2}\right|_* = \frac{a^2+16\alpha\sin^2(\theta_*/2)\cos^2(\theta_*/2)-4\alpha\cos^4(\theta_*/2)}{4\cos^2(\theta_*/2)} = 4\alpha\sin^2(\theta_*/2)$$

Hence, taking account of the relation defining θ_*, we find the frequency

$$\omega_* = [4\alpha(4\alpha-a^2)]^{1/2}$$

of small vibrations in the neighbourhood of the equilibria θ_*.

It follows from Theorem 3 that, in the non-resonance case, a unique 2π-periodic motion with an "amplitude" β exists in the neighbourhood of each of the equilibria $\pm\theta_*$. In the resonance case, one or three 2π-periodic motions exist with an "amplitude" $\beta^{1/3}$. This follows from Theorem 5 for, writing down Eqs (9.2) in the neighbourhood of the equilibria $\pm\theta_*$ in the form of system (5.1), we obtain in (5.4)

$$X_*\equiv 0,\quad Y_* = -(\beta/\omega_*)\sin\theta_*\cos\tau$$

In the classical case, the equilibria $\pm\theta_*$ of this system with one degree of freedom corresponds to a regular precession of a Lagrangian gyroscope. Hence, in the case of small vibrations of the suspension point, the periodic motions established above turn out to be pseudoregular precessions. In the resonance case, there can be one or three such motions. This fact was established earlier in the case when $|a| \neq |b|$ [14].

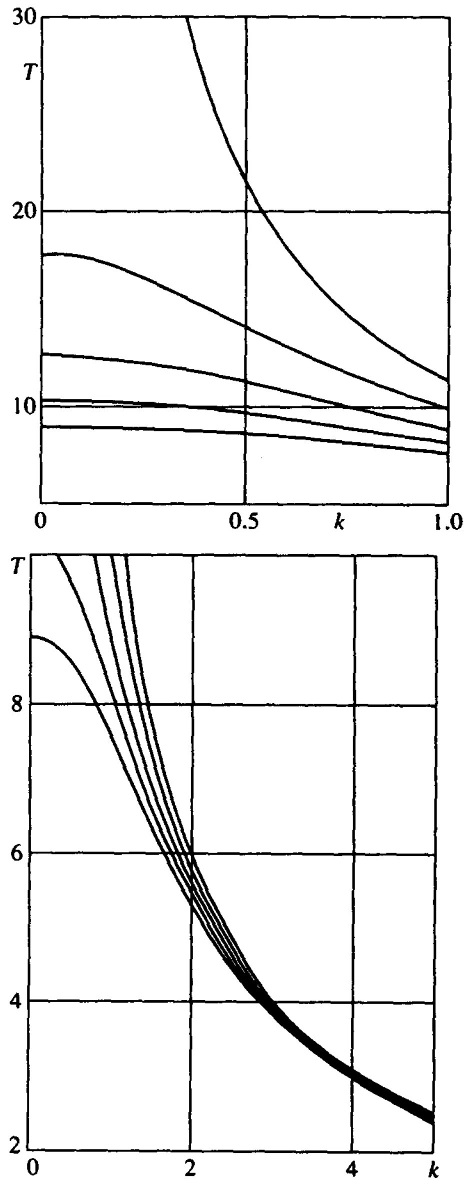

Fig. 4

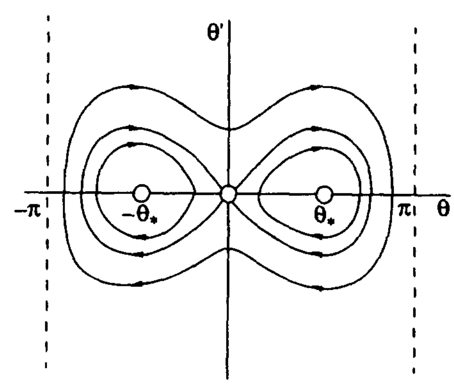

Fig. 5

The case when $a = -b$. The equation of the reduced system has the form

$$\theta'' - \frac{a^2 \operatorname{ctg}(\theta/2)}{2\sin^2(\theta/2)} + (-\alpha + \beta\cos\tau)\sin\theta = 0 \tag{9.6}$$

When there are no vibrations of the suspension point, we have a unique stable equilibrium position $\theta_* = \pi$ which corresponds to a "sleeping" Lagrangian gyroscope.

Vibrations about the equilibrium position occur with a period

$$T_* = 4\int_{\pi}^{\theta_0} \frac{d\theta}{\sqrt{2[h - W(\theta)]}} = 4\int_{0}^{\nu_0} \frac{d\nu}{\sqrt{2[h - W(\pi + \nu)]}}, \quad h = W(\theta_0)$$

and the frequency of small vibrations is equal to $\omega_* = (\alpha + a^2/4)^{1/2}$. If it is taken into account that

$$W(\pi + \nu) = (a^2/2)\operatorname{tg}^2(\nu/2) - \alpha\cos\nu$$

then we obtain that the period is calculated using formula (9.4) with just the replacement of α by $-\alpha$.

In the function $T_*(k, \gamma)$, negative values of the parameter γ correspond to negative α. The relations $T(k) = aT_*(k, \gamma)$, when $\gamma < 0$, are also shown in Fig. 3. It is seen that $dT_* \neq 0$. This guarantees the existence of 2π-periodic motions in the case of small vibrations of the suspension point. The "amplitudes" for these motions satisfy condition (9.5). We draw attention to the interesting fact that 2π-periodic motions exist in which the angle θ varies over the range $(\pi - \nu_0, \pi + \nu_0)$, $\nu_0 = 110° - 130°$.

I wish to thank V. V. Rumyantsev for his constant interest in and support of investigations into the vibrations of reversible mechanical systems.

This research was supported financially by the Russian Foundation of Basic Research (00-15-96150, 00-01-00122), the programme "Universities of Russia – Basic Research" (015.04.01.56) and the Federal special-purpose "Integration" Programme (2.1-294).

REFERENCES

1. DEVANEY, R. L., Reversible diffeomorphisms and flows. *Trans. Am. Math. Soc.*, 1976, **218**, 89–113.
2. KATS, A. M., Forced vibrations of non-linear systems with one degree of freedom which are close to conservative systems. *Prikl. Mat. Mekh.*, 1955, **19**, 1, 13–32.
3. TKHAI, V. N., Rotational motions of mechanical systems. *Prikl. Mat. Mekh.*, 1999, **63**, 2, 179–195.
4. MALKIN, I. G., *Some Problems in the Theory of Non-Linear Oscillations*. Gostekhizdat, Moscow, 1956.
5. BRYUNO, A. D., Analytical form of ordinary differential equations. *Tr. Mosk. Mat. Obshch.*, 1971, **25**, 119–262; 1972, **26**, 199–239.
6. SEVRYUK, M. B., Reversible systems. *Lecture Notes in Mathematics*. Berlin, Springer, 1986, 1211, 319 p.
7. TKHAI, V. N., Lyapunov families of motions in a reversible system. *Prikl. Mat. Mekh.*, 2000, **64**, 1, 46–58.
8. TKHAI, V. N., Reversibility of mechanical systems. *Prikl. Mat. Mekh.*, 1991, **55**, 4, 578–586.
9. TKHAI, V. N., Reversible mechanical systems. In *The Second Polyakhov Lectures Selected Papers*. Izd. Nauchn. Issled. Inst Khimii St Peterburg Univ., St Peterburg, 2000, 115–127.
10. TKHAI, V. N., The Lyapunov-Poincaré method in the theory of periodic motions. *Prikl. Mat. Mekh.*, 1998, **62**, 3, 355–371.
11. TKHAI, V. N., Non-linear oscillations of reversible systems. *Prikl. Mat. Mekh.*, 1995, **59**, 1, 38–50.
12. TKHAI, V. N., The stability of regular Grioli precessions. *Prikl. Mat. Mekh.*, 2000, **64**, 5, 848–857.
13. GRODMAN, D. L. and TKHAI, V. N., Rotations of a dynamically symmetric satellite under the action of gravitational forces and light pressure. In: *Modelling and Investigation of Complex Systems*. Mask. Gos. Akad. Priborostr. i inform., Izd. Moscow, Part. 3, 1998, 376–385.
14. KHOLOSTOVA, O. V., The dynamics of a Lagrangian gyroscope with a vibrating suspension point. *Prikl. Mat. Mekh.*, 1999, **63**, 5, 785–800.
15. KHOLOSTOVA, O. V., The stability of a "sleeping" Lagrangian gyroscope with a vibrating suspension point. *Prikl. Mat. Mekh.*, 2000, **64**, 5, 858–868.

Translated by E.L.S.